OPTIMAL BONDING OF PLATES IN THE PLANE STRESS STATE

S. B. Bushmanov and Yu. V. Nemirovskii

UDC 539.3:678.067.5

In the solution of optimization problems of the internal structure of elastic composites in the plane stress state, these latter were considered either as homogeneous anisotropic [1] or as a filamentary continuum [2, 3]. The condition of equal intensity of the armature fibers was the fundamental criterion for optimality in [4, 5].

On the basis of a model of a bonded layer proposed in [6], the problem of selecting the bonding direction and intensity corresponding to the minimum total armature volume in elastic plates loaded in their plane is examined in this paper. It is shown that designs with an equally stressed armature whose bonding directions are simultaneously the directions of the principal elongations will be optimal in the sense mentioned. Equations and boundary conditions are obtained that determine the bonding parameters of the optimal design. A relation is established between the optimal bonding trajectories and the slip lines for plane strain of a rigidly plastic body.

1. A plate is considered that consists of an isotropic matrix and a thin-fiber armature inserted in it, which is laid out in two directions forming the angles α_k with the positive direction 1 of a certain orthogonal coordinate system x_1, x_2. It is assumed that the plate has a constant unit thickness, is loaded in its plane by forces p_i on a part of the contour L_p, and is clamped rigidly on the remaining part of the contour L_u. There are no bulk forces. Both phases of the composite are considered linearly elastic, where the armature stiffness is considerably above the stiffness of the matrix. The bonding directions and intensities can vary independently.

The following relationships are taken as the mechanical model of the composite, where the averaged stresses σ_{ij}^c are connected to the structural stresses σ_{ij} in the matrix, σ_k in the armature, and the directions α_k and intensities ω_k of the bonding [6]:

$$\sigma_{ij}^c = (1-\omega)\,\sigma_{ij} + \omega_k \sigma_k l_{ik} l_{jk} \quad (i, j, k = 1, 2), \tag{1.1}$$

where $\omega = \omega_1 + \omega_2$; $l_{1k} = \cos\alpha_k$; $l_{2k} = \sin\alpha_k$. Here and henceforth, summation is over repeated subscripts.

The bonding intensities should satisfy the natural constraints

$$\omega_k \geqslant 0 \quad (k = 1, 2), \quad \omega \leqslant \omega_*, \tag{1.2}$$

where $\omega_* < 1$ is the ultimately achievable value of the total bonding intensity.

The structural stresses are related to the strains ε_{ij} of the composite by Hooke's law

$$\sigma_{11} = \frac{E_m}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}), \quad \sigma_{12} = \frac{E_m}{1+\nu}\varepsilon_{12}, \tag{1.3}$$

$$\sigma_{22} = \frac{E_m}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}); \tag{1.4}$$
$$\sigma_k = E_a \varepsilon_k = E_a \varepsilon_{ij} l_{ik} l_{jk} \quad (i, j, k = 1, 2).$$

Here E_a, E_m are the Young's modulus of the armature and matrix, and ν is the Poisson ratio of the matrix.

Let us consider the problem of determining the scheme for stowing the armature in the plate, which will correspond the the minimum total armature volume among all allowable designs. All possible plate designs satisfying the same boundary conditions and having an elastic pliability value measured as the work of the external forces on the displacements they produce, which do not exceed a given value J_0 are allowed in the comparison:

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 158-165, September-October, 1983. Original article submitted July 20, 1982.

0021-8944/83/2405-0755$07.50

$$J = \int_{L_p} p_i u_i dl \leqslant J_0. \tag{1.5}$$

Such designs will later be called allowable.

Let us show that among all allowable designs of a plate bonded by two families of fibers, the least total armature volume will be possessed by designs with an equally stressed armature laid out along the trajectories of the principal elongations of the composite, and having the limit value J_0 for the elastic pliability. Such designs will later be called optimal. Going from integration over the contour to integration over the domain S occupied by the middle plane of the plate by using Green's formula in (1.5), and using the relationship (1.1) here, the elastic pliability of the optimal and the arbitrarily allowed designs can be represented in the form

$$J_0 = \iint_S (1-\omega)\,\sigma_{ij}\varepsilon_{ij}ds + \iint_S \omega_k\sigma_k\varepsilon_k ds; \tag{1.6}$$

$$J^* = \iint_S (1-\omega^*)\,\sigma^*_{ij}\varepsilon^*_{ij}ds + \iint_S \omega^*_k\sigma^*_k\varepsilon^*_k ds \leqslant J_0. \tag{1.7}$$

Here and henceforth, all the quantities with asterisks correspond to the arbitrary allowable design, and without the asterisk, to the optimal design.

We now apply the principle of virtual work to the arbitrary allowable design by taking the real displacement field of the optimal design as the virtual displacement field:

$$J_0 = \iint_S (1-\omega^*)\,\sigma^*_{ij}\varepsilon_{ij}ds + \iint_S \omega^*_k\sigma^*_k\varepsilon'_k ds. \tag{1.8}$$

Here ε'_k is the virtual elongation in the directions of stowing the armature of the arbitrary allowable design, calculated according to the real displacement field of the optimal design.

By using the relationships (1.4), (1.6), (1.8) and the easily verifiable

$$\sigma_{ij}\varepsilon^*_{ij} = \sigma^*_{ij}\varepsilon_{ij}$$

the inequality (1.7) can be converted into the following:

$$E_a \iint_S \left(\omega^*_k - \omega_k\right)\varepsilon_k^2 ds - \iint_S (\omega^* - \omega)\,\sigma_{ij}\varepsilon_{ij}ds \geqslant \iint_S (1-\omega^*)\left(\sigma^*_{ij} - \right. \tag{1.9}$$
$$\left. - \sigma_{ij}\right)\left(\varepsilon^*_{ij} - \varepsilon_{ij}\right)ds + E_a \iint_S \omega^*_k\left(\varepsilon^*_k - \varepsilon'_k\right)^2 ds + E_a \iint_S \omega^*_k\left(\varepsilon_k^2 - \varepsilon'^2_k\right)ds.$$

Since the difference in the stresses $\sigma^*_{ij} - \sigma_{ij}$ and strains $\varepsilon^*_{ij} - \varepsilon_{ij}$ are related by Hooke's law (1.3), while the bonding intensities of any allowable design satisfy the constraints (1.2), then under the additional condition $\varepsilon_1^2 = \varepsilon_2^2$ all the components in the right side of (1.9) are nonnegative. Since the armature in the optimal design is equally stressed and laid out along the trajectories of the principal elongations, then taking (1.3) into account, the inequality (1.9) becomes

$$(1 - E_k)\,\varepsilon_1^2\left(V^*_a - V_a\right) \geqslant 0 \quad (k = 1, 2),$$

where $E_{1,2} = 2E_m/(1 \pm \nu)E_a$.

Here V_a and V^*_a are the total armature objects of the optimal and arbitrary allowable designs, E_1 corresponds to optimal designs with armature elongations $\varepsilon_1 = -\varepsilon_2$ (optimal designs of the first kind), and E_2 corresponds to designs with armature elongations $\varepsilon_1 = \varepsilon_2$ (optimal designs of the second kind).

The condition $E_k < 1$ therefore assures that optimal designs possess plates of minimal armature volume among all the allowable designs. Since the optimal designs of the second kind are in a homogeneous deformed state, they can only be realized in cases when the boundary conditions on the plate contour are given in stresses. As will be shown below, the condition of rigid clamping of the plate on the contour L_u permits the unique construction of optimal designs of the first kind.

2. Since the bonding directions of an optimal design of the first kind are mutually

orthogonal, then it is possible to set $\alpha_1 = \alpha_2 - \pi/2 = \alpha$. Let u, v be the displacement components in the directions of the x, y axes of a Cartesian rectangular coordinate system. Then the strain and rotation components of an optimal design of the first kind can be represented in the form

$$\begin{gathered}\varepsilon_x = \partial u/\partial x = \varepsilon_1 \cos 2\alpha, \ \varepsilon_y = \partial v/\partial y = -\varepsilon_1 \cos 2\alpha, \\ \varepsilon_{xy} = (1/2)(\partial v/\partial x + \partial u/\partial y) = \varepsilon_1 \sin 2\alpha, \ \Omega = (1/2)(\partial v/\partial x - \partial u/\partial y).\end{gathered} \tag{2.1}$$

Eliminating u and v from (2.1) by using cross differentiation, we obtain a system of equations governing the bonding directions of the optimal design of the first kind:

$$\begin{gathered}\partial\Omega/\partial x - 2\varepsilon_1(\cos 2\alpha\partial\alpha/\partial x + \sin 2\alpha\partial\alpha/\partial y) = 0, \\ \partial\Omega/\partial y - 2\varepsilon_1(\sin 2\alpha\partial\alpha/\partial x - \cos 2\alpha\partial\alpha/\partial y) = 0.\end{gathered} \tag{2.2}$$

The system (2.2) is hyperbolic, where its characteristic directions are in agreement with the bonding directions. As is easily noted, the system (2.2) is identical to the fundamental equations to the theory of plane strain of a rigidly plastic body [7]. Therefore, the bonding trajectories of an optimal design of the first kind agree geometrically with the slip lines of a certain Hencke—Prandtl grid, and possess all the known properties of these latter. The most essential of these properties are formulated in Hencke theorems [7]. Analogs of the first and second Hencke theorems are the following assertions: 1) Upon going from one fiber to another of the same family along any fiber of the other family, the angle α varies by the very same quantity; 2) for motion along a fixed fiber the radii of curvature of fibers of the other family vary within the distance traversed. On the basis of the formulated theorems, the system (2.2) can be replaced by another system more convenient for numerical computations [8]:

$$dR_2 + R_1 d\alpha = 0, \ dR_1 - R_2 d\alpha = 0. \tag{2.3}$$

Here R_1, R_2 are the radii of curvature of the first and second family fibers. Equations (2.3) are valid along the first and second family fibers, respectively.

If the angle α is determined, then the bonding trajectories are found from solution of the equations

$$dy/dx = \operatorname{tg}\alpha, \ dy/dx = -\operatorname{ctg}\alpha, \tag{2.4}$$

which are valid along the first and second family fibers, respectively. By introducing the auxiliary variables

$$\bar{x} = x\cos\alpha + y\sin\alpha, \ \bar{y} = -x\sin\alpha + y\cos\alpha \tag{2.5}$$

the equations (2.4) can be represented in a more convenient form for numerical computations

$$d\bar{y} + \bar{x}d\alpha = 0, \ d\bar{x} - \bar{y}d\alpha = 0, \tag{2.6}$$

which are valid along the first and second family fibers, respectively.

The average stresses in the optimal design of the first kind can be represented by virtue of the relationships (1.1), (1.3), (1.4), and (2.1), in the form

$$\begin{gathered}\sigma_x^c = \frac{1}{2}\sigma_1(\chi + \varkappa\cos 2\alpha), \\ \sigma_y^c = \frac{1}{2}\sigma_1(\chi - \varkappa\cos 2\alpha), \quad \tau_{xy}^c = \frac{1}{2}\sigma_1\varkappa\sin 2\alpha,\end{gathered} \tag{2.7}$$

where

$$\chi = \omega_1 - \omega_2, \ \varkappa = E_1 + (1 - E_1)\omega. \tag{2.8}$$

Substituting (2.7) into the equilibrium equations

$$\frac{\partial\sigma_x^c}{\partial x} + \frac{\partial\tau_{xy}^c}{\partial y} = 0, \ \frac{\partial\tau_{xy}^c}{\partial x} + \frac{\partial\sigma_y^c}{\partial y} = 0 \tag{2.9}$$

results in the following system of equations

$$\frac{\partial\chi}{\partial x} + \frac{\partial}{\partial x}(\varkappa\cos 2\alpha) + \frac{\partial}{\partial y}(\varkappa\sin 2\alpha) = 0, \tag{2.10}$$

$$\frac{\partial \chi}{\partial y} + \frac{\partial}{\partial x}(\varkappa \sin 2\alpha) - \frac{\partial}{\partial y}(\varkappa \cos 2\alpha) = 0, \quad (2.10)$$

which is hyperbolic with characteristics that are in agreement with the bonding trajectories. For numerical computations, a more convenient form of (2.10) is written along the first and second family fibers, respectively

$$R_2 d(\chi + \varkappa) - 2\varkappa R_1 d\alpha = 0, \; R_1 d(\chi - \varkappa) - 2\varkappa R_2 d\alpha = 0. \quad (2.11)$$

On the basis of analogs of the Hencke theorems and the definitions of the bonding intensities [6], these latter can be represented in the form

$$\omega_1 = -AS_1/R_2, \; \omega_2 = -AS_2/R_1, \quad (2.12)$$

where S_1 and S_2 are the areas of the first and second family fiber cross-sections, and A is a positive constant whose value is not essential in the present consideration. Taking account of (2.8) and (2.12), the equations (2.11) can be transformed in the case $E_1 = 0$ into the form

$$dS_1 - S_2 d\alpha = 0, \; dS_2 + S_1 d\alpha = 0. \quad (2.13)$$

A deduction can be made from an analysis of Eqs. (2.13), which are valid along the first and second family fibers, respectively, that in the case $E_1 = 0$ the cross-sectional areas of the armature fibers decrease monotonically with motion along the fiber towards divergence. Because of the continuous dependence of the solution of the problem under consideration on the input data, the assertion formulated remains valid even for sufficiently small values of $E_1 > 0$.

The boundary conditions formulated at the beginning of Sec. 1 can be reformulated in terms of the functions introduced above. Because of the rigid clamping of the plate along the contour L_u, we obtain the following boundary conditions for the system (2.2) on this contour

$$\alpha = \varphi \pm \pi/4 + n\pi, \; \Omega = \pm \varepsilon_1, \quad (2.14)$$

where φ is the angle between the external normal to L_u and the x axis, and n is an arbitrary integer. If the boundary conditions (2.14) are compared with formulas (35.4) from [7], then it can be noted that the Cauchy problem for the system (2.2) with the boundary conditions (2.14) is completely analogous to the problem of plane strain for a rigidly plastic body with load-free boundaries of the same shape as the contour L_u. The field of slip lines of this problem, meaning the field of bonding trajectories of an optimal design of the first kind, can be constructed by well-known methods [7, 8]. Since only parameters of the contour L_u are in the boundary conditions (2.14), the bonding trajectories of the optimal design of the first kind depend only on the shape of L_u. The selection of the signs in (2.14) does not influence the geometry of the bonding trajectories, but only establishes which of the fiber families works under tension, and which under compression, and should be in agreement with the kind of loads applied to L_p. If L_u is a closed, smooth, convex contour, then as is shown in [9], the field of slip lines and perhaps even the field of bonding trajectories outside the contour L_u are formed by spiral curves that diverge with distance from L_u, and can be continued as far from L_u as desired. If it turns out here that the contour L_p on which the loads are given does not have the characteristic direction anywhere, and intersects each characteristic just once, then the Cauchy problem for the system (2.10) will be correct. Let the normal pressure σ and the tangential stress τ be given on the contour L_p; then by virtue of (2.7), the boundary conditions on the contour L_p can be written in the form

$$\sigma = (1/2)\sigma_1[\chi + \varkappa \cos 2(\alpha - \varphi)], \; \tau = (1/2)\sigma_1 \varkappa \sin 2(\alpha - \varphi), \quad (2.15)$$

where φ is the angle between the external normal to L_p and the x axis.

Therefore, the problem of construction an optimal design of the first kind is subdivided into two stages. The Cauchy problem for the system (2.2) with the boundary conditions (2.14) given on the contour L_u is solved in the first stage. The solution of this problem determines the field of bonding trajectories in the domain S. The Cauchy problem for the system (2.10) with the boundary conditions (2.15) given on the contour L_p is solved in the second stage for the bonding directions already known. This problem is solved in the direction opposite to the first stage. After it has been solved, all the parameters of the optimal design of the first kind become known. It should be kept in mind that the solution constructed in this manner has a mechanical meaning only when the inequalities (1.2) are satisfied, which impose definite constraints on the external loads σ, τ. These constraints depend in a substantial manner on the shape of the domain S.

The systems (2.3) and (2.6) are utilized for a numerical construction of the bonding trajectories fields. The appropriate values of the radii of fiber curvature R_1 and R_2 on the contour L_u are defined by the formulas

$$\frac{1}{R_1} = \pm\frac{\sqrt{2}}{2}\frac{\partial\varphi}{\partial l}, \quad \frac{1}{R_2} = -\frac{\sqrt{2}}{2}\frac{\partial\varphi}{\partial l},$$

in which e is the derivative taken with respect to the direction tangent to the contour L_u defined by the relationship

$$\partial/\partial l = -\sin\varphi\,\partial/\partial x + \cos\varphi\,\partial/\partial y.$$

The initial conditions for the system (2.6) are determined by the relationships (2.5) in which x, y are coordinates of points of the contour L_u while the angle α is determined from the boundary conditions (2.14).

3. As the simplest example, we construct an optimal design of the first kind for a circular annular plate fastened along the inner circle of radius r_0 and loaded uniformly by loads σ, τ distributed over the outer circle of radius r_1. In this case, two families of logarithmic spirals that intersect the radial directions at the angles $\pm\pi/4$ will be the trajectories of optimal armature layout. If we go over to dimensionless variables

$$R = r/r_0,\ R_0 = r_1/r_0,\ \sigma_R = \sigma_r^c/\sigma_1,\ \sigma_\theta = \sigma_\theta^c/\sigma_1,\ \tau_{R\theta} = \tau_{r\theta}^c/\sigma_1,$$

where $\sigma_r^c, \sigma_\theta^c, \tau_{r\theta}^c$ are the average stresses in a polar (r, θ) coordinate system, and if we consider the external loads σ, τ, also dimensionless in σ_1, then the bonding intensities of an optimal design of the first kind are determined by the following

$$\omega_{1,2} = \frac{1}{1-E_1}\left(|\tau|\frac{R_0^2}{R^2} - \frac{1}{2}E_1\right) \pm \sigma.$$

Since the radii of curvature of the logarithmic spirals are $R_1 = R_2 = -R\sqrt{2}$ then according to (2.12) the areas of the armature fiber cross-sections vary as follows:

$$S_{1,2} = \frac{A\sqrt{2}}{1-E_1}\left\{|\tau|\frac{R_0^2}{R} - \left[\frac{1}{2}E_1 \pm (1-E_1)\sigma\right]R\right\}.$$

The stress state of the design constructed has the form

$$\sigma_R = \sigma_\theta = \sigma, \quad \tau_{R\theta} = \tau R_0^2/R^2.$$

The inequalities (1.2) impose the following constraints on the external loads

$$|\tau| \pm (1-E_1)\sigma \geqslant \frac{1}{2}E_1, \quad |\tau| \leqslant [E_1 + (1-E_1)\omega_*]/2R_0^2. \tag{3.1}$$

The inequalities (3.1) determine the domain of allowable external load values in the plane σ, τ, which consists of two triangular parts and is symmetric relative to both axes. This domain is not empty when the following condition is satisfied

$$R_0^2 \leqslant 1 + \omega_*(1-E_1)/E_1,$$

which determines the maximally allowable relative dimensions of the plate. The total relative volume of the armature of the constructed optimal design of the first kind is

$$V_0 = \frac{1}{1-E_1}\left(\frac{4|\tau|R_0^2\ln R_0}{R_0^2-1} - E_1\right).$$

4. The method of characteristics with the constant increment $\Delta\alpha$ between adjacent nodes

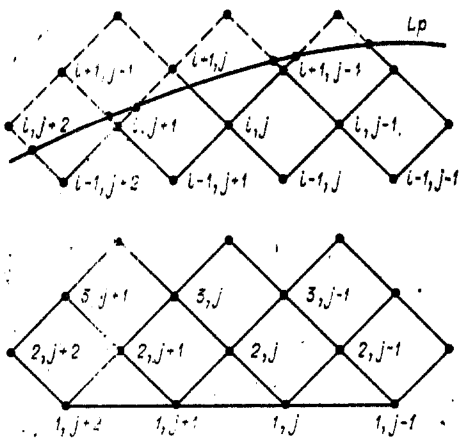

Fig. 1

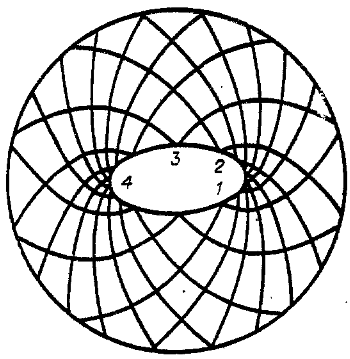

Fig. 2

[8] is utilized in the numerical construction of optimal designs of the first kind. The constancy of $\Delta\alpha$ over the whole field of bonding trajectories is assured by an analog of the first Hencke theorem formulated in Sec. 2. The differential equations (2.3), (2.6), and (2.11) are replaced by difference equations. For instance, the system (2.6) is replaced by the difference equations

$$y_{i+1,j} - y_{i,j+1} = (x_{i,j+1} + x_{i+1,j})\Delta\alpha/2, \qquad (4.1)$$
$$x_{i+1,j} - x_{i,j} = (y_{i,j} + y_{i+1,j})\Delta\alpha/2.$$

The following numbering of the mesh nodes is taken here: The first subscript denotes the number of the node layer, and the second the number of the node in the layer, where the numbering of the nodes in the adjacent layers is selected in such a manner that the nodes corresponding to one bonding trajectory of the second family have the very same number j. A characteristic fragment of the mesh is shown schematically in Fig. 1. The first layer of mesh nodes lies in the contour L_u. The coordinates of the remaining nodes are not known in advance and are determined during the computation.

The difference equations (4.1) approximate the initial differential equations (2.6) to second order accuracy in the mesh spacing $\Delta\alpha$. Solving (4.1) for the values of the mesh functions on the (i + 1)-th layer, we obtain the following recursion relations

$$x_{i+1,j} = A_1 x_{i,j} + A_2(y_{i,j} + y_{i,j+1}) + A_3 x_{i,j+1}, \qquad (4.2)$$
$$y_{i+1,j} = A_1 y_{i,j+1} + A_2(x_{i,j} + x_{i,j+1}) + A_3 y_{i,j},$$

where $A_1 = (1 - \Delta\alpha^2/4)^{-1}$; $A_2 = A_1\Delta\alpha/2$; $A_3 = A_2\Delta\alpha/2$.

The recursion relations (4.2) permit approximate construction of the bonding trajectory mesh for an optimal design of the first kind. Recursion relations to determine the radii of fiber curvature and the bonding intensities are obtained in an analogous manner, and not presented here in the interests of saving space.

Nodes of the bonding trajectory mesh constructed by using the recursion relations (4.2) do not generally fall in the contour L_p (see Fig. 1). Hence, the following procedure is proposed for approximate assignment of the initial values χ and $\varkappa$: if two adjacent mesh nodes belonging to the very same bonding trajectory lie on different sides of the contour L_p, then the point of intersection between this bonding trajectory and the contour L_p is found by using inverse linear interpolation. Considering this point an additional mesh node, we determine values of the fiber radii of curvature there and the increment in the angle α when going from the interior node to the additional node by using linear interpolation. Then by using the boundary conditions (2.15) we determine the values of χ and $\varkappa$ at the additional node.

The numerical algorithm described above was used to construct an optimal design of the first kind for a circular plate with an elliptical hole. The plate was loaded by uniformly distributed loads σ, τ along the outer contour L_p and was rigidly clamped along the inner

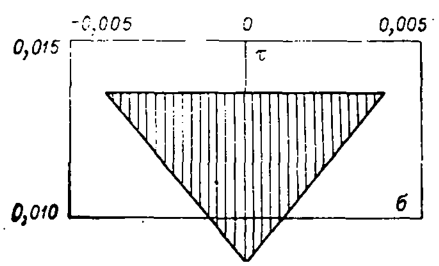

Fig. 3

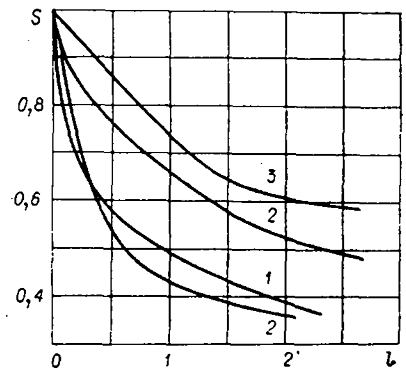

Fig. 4

elliptical contour L_u. Computations were performed on a BÉSM-6 computer with mesh spacing $\Delta\alpha = \pi/64$. The mesh for bonding trajectories of an optimal plate design constructed by using a graph-plotter is shown in Fig. 2; the ratio of the L_u contour axes is 2 while the ratio between the radius of the contour L_p and the major semiaxis of the contour L_u equals 2.5. Only each quarter of the mesh lines constructed during the computation is shown here. The domain of allowable values of the external loads on the (σ, τ) plane, as constructed for $\nu = 0.4$, $E_m/E_a = 0.01$, and the maximally allowable total bonding density $\omega_* = 0.8$, is represented in Fig. 3. This domain corresponds to designs in which the tensile fibers are warped counterclockwise. The domain of allowable loads for designs in which the tensile fibers are warped clockwise is symmetric with respect to the σ axis to the domain presented in Fig. 3. Graphs of the change in fiber cross-section area are presented in Fig. 4 as a function of their running length, as constructed for the external loads $\sigma = 0$, $\tau = 0.012$. It was assumed here that the cross-sectional area equals 1 at the initial point on the contour L_u. Curves 1-4 in Fig. 4 correspond to fibers emerging from the points 1-4 on the contour L_u (see Fig. 2) and being warped counterclockwise.

5. Since the strain state of optimal designs of the second kind is homogeneous, the equation of strain compatibility is satisfied automatically without imposing any constraints on the bonding parameters. To determine the four structural parameters of optimal designs of the second kind, only the two equilibrium equations (2.9) remain, which after the relationships (1.1) with the Hook's law (1.3), (1.4) and the homogeneity of the strain state ($\varepsilon_x = \varepsilon_y = \varepsilon_1 = \varepsilon_2$, $\varepsilon_{xy} = 0$) taken into account have been substituted, take the form

$$\frac{\partial}{\partial x}\left(\omega_1\cos^2\alpha_1\right)+\frac{\partial}{\partial y}\left(\omega_1\sin\alpha_1\cos\alpha_1\right)+\frac{\partial}{\partial x}\left(\omega_2\cos^2\alpha_2\right)+\frac{\partial}{\partial y}\left(\omega_2\sin\alpha_2\cos\alpha_2\right)-E_2\frac{\partial\omega}{\partial x}=0,$$
$$\frac{\partial}{\partial x}\left(\omega_1\sin\alpha_1\cos\alpha_1\right)+\frac{\partial}{\partial y}\left(\omega_1\sin^2\alpha_1\right)+\frac{\partial}{\partial x}\left(\omega_2\sin\alpha_2\cos\alpha_2\right)+\frac{\partial}{\partial y}\left(\omega_2\sin^2\alpha_2\right)-E_2\frac{\partial\omega}{\partial y}=0. \qquad (5.1)$$

Equations (5.1) determine the class of designs that are equivalent in the sense of equality of the total armature volume. Since the system (5.1) is not closed, infinitely many such designs can be constructed for given boundary conditions. Conditions for the technological efficiency in fabrication are drawn upon to extract specific and most preferable designs. The most essential are requirements about constancy of the cross-sectional area of each individual armature fiber. To establish a relationship between the bonding parameters that exist in this case, the vector fields $\boldsymbol{\omega}_k$ with the components $\omega_k l_{ik}$($i, k = 1, 2$, no summation over k) are introduced into the considerations. Let $D \subset S$ be an arbitrary simply-connected domain with the smooth boundary Γ, while $\mathbf{n}$ is the external normal vector to Γ. Then the absolute value of the scalar product $\boldsymbol{\omega}_k\mathbf{n}$ has the meaning of the total cross-sectional area of the k-th family fibers passing through unit arclength of the contour Γ. Here $\boldsymbol{\omega}_k\mathbf{n}$ is negative if the fibers enter the domain D and positive if the fibers emerge from the domain D. Since the fibers have constant cross-section and cannot be broken off within the domain D, then the total cross-sectional area of the fibers entering D equals the total cross-sectional area of the fibers emerging from D, i.e.,

$$\int_\Gamma \boldsymbol{\omega}_k\mathbf{n}\,dl = \int\int_D \operatorname{div}\boldsymbol{\omega}_k\,ds = 0 \quad (k = 1, 2). \qquad (5.2)$$

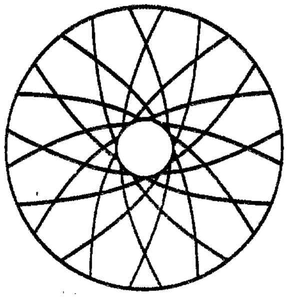

Fig. 5

Here the Gauss—Ostrogradskii formula is used to go from integration over the contour Γ to integration over the domain D. Because of the arbitrariness of the domain D it follows from (5.2) that

$$\frac{\partial}{\partial x}(\omega_k \cos\alpha_k) + \frac{\partial}{\partial y}(\omega_k \sin\alpha_k) = 0 \quad (k = 1, 2). \tag{5.3}$$

If the operators of differentiation with respect to the directions parallel and orthogonal to the k-th family fibers are introduced

$$\partial/\partial l_k = \cos\alpha_k \partial/\partial x + \sin\alpha_k \partial/\partial y, \ \partial/\partial n_k = -\sin\alpha_k \partial/\partial x + \cos\alpha_k \partial/\partial y,$$

then the system (5.1), (5.3) can be written in the form

$$\begin{gathered}\omega_1 \sin\alpha\partial\alpha_1/\partial l_1 + E_2\partial\omega/\partial l_2 = 0, \ \partial\omega_1/\partial l_1 + \omega_1\partial\alpha_1/\partial n_1 = 0, \\ \omega_2 \sin\alpha\partial\alpha_2/\partial l_2 - E_2\partial\omega/\partial l_1 = 0, \ \partial\omega_2/\partial l_2 + \omega_2\partial\alpha_2/\partial n_2 = 0,\end{gathered} \tag{5.4}$$

where $\alpha = \alpha_1 - \alpha_2$.

The system (5.4) belongs to the elliptic type and determines the bonding directions and intensities of optimal designs of the second kind with fibers of constant cross-section.

We present the bonding trajectories of a circular annular plate with relative dimension $R_0 = 5$, loaded by normal loads p and q distributed uniformly over the outer and inner contours as the simplest example of an optimal design of the second kind. Omitting the details, we just note that the bonding directions are determined in this case from the solutions of the equation

$$(w + E_2)dw/dR + (2w/R)(w - 1 + E_2) = 0, \tag{5.5}$$

where $w = \cos^2\alpha$; α is the angle between the fibers and the radial directions. Solutions of (5.5) emerge rapidly onto the asymptotic regime $w = 1 - E_2$ as R grows, i.e., for sufficient distance from the inner contour of the plate the bonding trajectories asymptotically approach logarithmic spirals that intersect the radial directions at the angles $\pm$ arccos $\sqrt{1-E_2}$. Bonding trajectories are shown in Fig. 5 for an optimal design of the second kind constructed by a numerical method for the following values of the input parameters $E_2 = 0.1$, $q = 0.2$, and $p = 0.15$.

LITERATURE CITED

1. N. V. Banichuk, "Optimization of anisotropic properties of deformable media and plane problems of elasticity theory," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 1 (1979).
2. G. I. Bryzgalin, "On a rational design of anisotropic plane bodies with weak binder," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 4 (1969).
3. G. I. Bryzgalin, "Optimal design of locally orthotropic plane elastic bodies with weak binder," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 3 (1971).
4. Yu. A. Bogan and Yu. V. Nemirovskii, "On certain problems of optimal control for a bonded medium," in: Applied Strength and Plasticity Problems [in Russian], No. 1, Izd. Gor'k. Univ., Gor'kii (1975).
5. Yu. V. Nemirovskii, "On the question of optimal stowing of an armature in plates," Mekh. Polim., No. 4 (1978).
6. Yu. V. Nemirovskii, "On the elastic-plastic behavior of a bonded layer," Prikl. Mekh. Tekh. Fiz., No. 6 (1969).

7. L. M. Kachanov, Principles of the Theory of Plasticity [in Russian], Nauka, Moscow (1969).
8. R. Hill, Mathematical Theory of Plasticity, Oxford Univ. Press (1950).
9. S. A. Khristianovich, "Plane problem of the mathematical theory of plasticity for external forces given on a closed contour," Mat. Sbornik, Nov. Ser. 1(43), No. 4 (1936).